

Use of Complex Lie Symmetries for Linearization of Systems of Differential Equations - II: Partial Differential Equations

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February 2, 2008

Abstract

The linearization of complex ordinary differential equations is studied by extending Lie's criteria for linearizability to complex functions of complex variables. It is shown that the linearization of complex ordinary differential equations implies the linearizability of systems of partial differential equations corresponding to those complex ordinary differential equations. The invertible complex transformations can be used to obtain invertible real transformations that map a system of nonlinear partial differential equations into a system of linear partial differential equation. Explicit invariant criteria are given that provide procedures for writing down the solutions of the linearized equations. A few non-trivial examples are mentioned.

1 Introduction

The linearization i.e. use of point transformations to convert the nonlinear equations to linear form of real ordinary differential equations (RODEs) was first introduced by Lie [7]. He provided the linearizability criteria for any scalar second order RODE to be point transformable to a linear RODE via invertible maps of both independent and dependent variables. He found that the RODE must be at most cubic in the first derivative and that a particular over-determined system of conditions for the coefficients must be satisfied (see e.g., [10 – 13], [15]). Tresse [25] studied the linearization of a second order RODE by looking at the two relative invariants of the equivalence group of transformations, the vanishing of both of which gives the necessary and sufficient conditions for linearization. These are equivalent to the Lie conditions [15].

Lie's approach has been extended in many ways. The linearization of higher order RODEs and their systems has been discussed by various authors (see e.g., [15, 19, 24, 27]). Further, linearization of PDEs is discussed in [5, 6, 13]. It is seen that for systems of RODEs, Lie linearizability criteria involves solving more than ten equations simultaneously that become complicated and messy. Also, for PDEs and their systems it is highly nontrivial to obtain the transformations that remove nonlinearity in the equations. In a companion paper, henceforth called P-I [3], the Lie linearizability criteria is extended to complex functions of real variables to obtain linearizability criteria for systems of RODEs.

We have extended the Lie approach of linearization of RODEs via complex point transformations to the complex domain and obtained corresponding Lie criteria for complex ordinary differential equations (CODEs). A CODE is a differential equation of a single complex function of one complex variable. The CODEs can be linearized in the same way as RODEs via complex point transformations. A CODE yields a system of partial differential equation (PDEs) after decomposing all the complex functions, variables and derivatives into their real and imaginary parts. That also yield Cauchy-Riemann (CR) equations which are already linear thus yield no difference for the linearization of systems of PDEs. The linearization of systems of PDEs corresponding to a CODE follows directly from the linearization of that CODE via real point transformations that are obtained after decomposing the complex point transformation. This is an intrinsic property of complex theory, which Penrose calls “complex magic” [23], that yields a non-trivial way of

linearizing PDEs and their systems.

The outline of the paper is as follows. In the second section we discuss those CODEs that are equivalent, i.e. which can be transformed into one another via invertible complex mappings. Lie compatibility conditions and some other results for linearization of systems of PDEs corresponding to CODEs are discussed in the third section. Examples are also discussed in the same section. Finally, in the fourth section we have given a summary and discussion regarding our complex Lie symmetry (CLS) analysis for linearization.

2 Equivalent CODEs

Two CODEs are (locally) equivalent via an invertible complex transformation if one can be transformed into the other by an invertible complex transformation. A CODE is a system of four PDEs, which includes the two linear CR equations in two unknown functions of two independent variables. If a CODE is equivalent to another CODE via invertible complex transformations then the system of PDEs corresponding to that CODE is also equivalent to the other system of PDEs. The invertible complex transformation yields two real transformations, which can be used to transform one system of PDEs to another system.

Every first-order CODE

$$u'(z) = w(z, u), \quad (1)$$

can be transformed into the simplest one $u' = 0$, via a complex point transformation [16]. Similarly, every linear second-order CODE can always be transformed into its simplest form via invertible complex transformations, specifically into $u''(z) = 0$. For example, the complexified Ricatti equation

$$u'(z) + u^2 = 0, \quad (2)$$

is transformable to $U' = 0$ by means of

$$Z = z, \quad U = (1/u) - z. \quad (3)$$

Also, the complexified simple harmonic oscillator equation

$$u''(z) + u = 0, \quad (4)$$

can be transformed into $U'' = 0$ via an invertible complex transformation

$$Z = \tan z, \quad U = u \sec z. \quad (5)$$

To decompose the above CODEs into corresponding systems of PDEs, write

$$z = x + iy, \quad u(z) = f(x, y) + ig(x, y), \quad (6)$$

$$w = w_1 + iw_2. \quad (7)$$

The system of PDEs corresponding to a first-order CODE is

$$f_x + g_y = w_1(x, y, f, g), \quad g_x - f_y = w_2(x, y, f, g), \quad (8)$$

which can be transformed into

$$F_x + G_y = 0, \quad G_x - F_y = 0, \quad (9)$$

by an invertible real transformation derived from the complex point transformation. Similarly, the system of PDEs corresponding to a linear second-order CODE is

$$f_{xx} - f_{yy} + 2g_{xy} = w_1(x, y, f, g, h, l), \quad (10)$$

$$g_{xx} - g_{yy} - 2f_{xy} = w_2(x, y, f, g, h, l), \quad (11)$$

where both w_1 and w_2 are two such real functions that do not give rise to a nonlinear system. The above system can be transformed into

$$F_{XX} - F_{YY} + 2G_{XY} = 0, \quad (12)$$

$$G_{XX} - G_{YY} - 2F_{XY} = 0. \quad (13)$$

via an invertible real transformation that is obtained from a complex transformation.

Examples

1. The complexified Riccati equation is equivalent to

$$f_x + g_y = -f^2 + g^2, \quad (14)$$

$$g_x - f_y = -2fg. \quad (15)$$

If we apply the following transformation

$$X = x, \quad Y = y, \quad (16)$$

$$F = \frac{f}{f^2 + g^2} - x, \quad G = \frac{-g}{f^2 + g^2} - y, \quad (17)$$

we get

$$F_X + G_Y = 0, \quad (18)$$

$$G_X - F_Y = 0. \quad (19)$$

Notice that a simple complex transformation for the complexified Riccati equation yields a non-trivial real transformation (16) and (17) that transforms the system of PDEs (14) and (15) into its simple analogue (18) and (19). This remarkable feature of complex theory is of great significance for the linearization of PDEs and their systems.

2. The system

$$f_{xx} - f_{yy} + 2g_{xy} = -f, \quad (20)$$

$$g_{xx} - g_{yy} - 2f_{xy} = -g, \quad (21)$$

can be transformed into

$$F_{XX} - F_{YY} + 2G_{XY} = 0, \quad (22)$$

$$G_{XX} - G_{YY} - 2F_{XY} = 0, \quad (23)$$

via the real transformations

$$X = \frac{(1/2) \sin 2x}{\cos^2 x + \sinh^2 y}, \quad Y = \frac{(1/2) \sinh 2y}{\cos^2 x + \sinh^2 y}, \quad (24)$$

$$F = \frac{f \cos x \cosh y - g \sin x \sinh y}{\cos^2 x + \sinh^2 y}, \quad G = \frac{f \sin x \sin y + g \cos x \cosh y}{\cos^2 x + \sinh^2 y}. \quad (25)$$

Thus, the system of PDEs corresponding to a CODE can easily be transformed into its simplified (linear) form by invoking complex transformations. It would have been very difficult to guess or calculate the real transformations (24) and (25) that transform system (20) and (21) into system (22) and (23) without this formalism. It is an extraordinary characteristic of complex magic.

Lie conditions in the complex domain can be derived by replacing the real variable x in r-CODE in P-I with a complex variable z . A scalar second order CODE which is at most cubic in its first derivative

$$u''(z) = A(z, u)u'^3 + B(z, u)u'^2 + C(z, u)u' + D(z, u), \quad (26)$$

where A , B , C and D are complex valued functions, is linearizable according to Lie theorem. We do not re-state Theorem 1 of P-I as all the conditions (1-9) remain the same with only the change that the real variable x is replaced by a complex variable z . In the subsequent section we will decompose (26) into system of PDEs by using (6) and by taking h and l as real and imaginary parts of u' .

3 Lie Conditions for Systems of PDEs

According to Lie's theorem the necessary condition for a second order CODE to be linearizable is that it is at most cubic in its first order derivative implies the necessary conditions for system of PDEs corresponding to that CODE i.e. the system must be at most cubic in first derivatives together with certain constraints on coefficients. The real transformations for linearization of a system of nonlinear PDEs can be obtained by decomposing complex transformations that linearize a CODE. The general form of a system of PDEs corresponding to (26) is given by

$$f_{xx} - f_{yy} + 2g_{xy} = A^1(h^3 - 3hl^2) - A^2(3h^2l - l^3) + B^1(h^2 - l^2) - 2B^2hl + C^1h - C^2L + D^1, \quad (27)$$

$$g_{xx} - g_{yy} - 2f_{xy} = A^1(3h^2l - l^3) + A^2(h^3 - 3hl^2) + 2B^1hl + B^2(h^2 - l^2) + C^2h + C^1L + D^2. \quad (28)$$

where all the coefficients A^i , B^i , C^i and D^i are functions of x, y, f, g .

Now we derive Lie conditions for systems of PDEs corresponding to a second order CODE by decomposing results in Theorem 1 in P-I into real variables by invoking (37) and (38) and by taking ρ as a complex function of z .

Theorem 1. The following statements are equivalent.

1. The system of PDEs (27) and (28) is linearizable via real transformations;

2. The coefficients in (27) and (28) satisfy

$$\begin{aligned}
& 3A_{xx}^1 - 3A_{yy}^1 + 6A_{xy}^2 + 3C^1A_x^1 + 3C^1A_y^2 - 3A_x^2C^2 + 3C^2A_y^1 - 3A_f^1D^1 - 3D^1A_g^2 + \\
& 3D^2A_f^2 - 3D^2A_g^1 + 3A^1C_x^1 + 3A^1C_y^2 - 3A^2C_y^2 + 3A^2C_x^2 + C_{ff}^1 - C_{gg}^1 + 2C_{fg}^2 - \\
& 6A^1D_f^1 - 6A^1D_g^2 + 6A^2D_f^2 - 6A^2D_g^1 + B^1C_f^1 + B^1C_g^2 - B^2C_f^2 + B^2C_g^1 - \\
& 2B^1B_x^1 - 2B^1B_y^2 + 2B^2B_x^2 - 2B^2B_y^1 - 2B_{xf}^1 - 2B_{yf}^2 - 2B_{xg}^2 + 2B_{yg}^1 = 0,
\end{aligned} \tag{29}$$

$$\begin{aligned}
& 3A_{xx}^2 - 3A_{yy}^2 - 6A_{xy}^1 + 3C^2A_x^1 + 3C^2A_y^2 + 3A_x^2C^1 - 3C^1A_y^1 - 3D^2A_f^1 - 3D^2A_g^2 - \\
& 3D^1A_f^2 + 3D^1A_g^1 + 3A^2C_x^1 + 3A^2C_y^2 + 3A^1C_y^2 - 3A^1C_x^1 + C_{ff}^2 - C_{gg}^2 - 2C_{fg}^1 - \\
& 6A^2D_f^1 - 6A^2D_g^2 - 6A^1D_f^2 + 6A^1D_g^1 + B^2C_f^1 + B^2C_g^2 + B^1C_f^2 - B^1C_g^1 - \\
& 2B^2B_x^1 - 2B^2B_y^2 - 2B^1B_x^2 + 2B^1B_y^1 - 2B_{xf}^2 + 2B_{yf}^1 + 2B_{xg}^1 - 2B_{yg}^2 = 0
\end{aligned} \tag{30}$$

$$\begin{aligned}
& 6D^1A_x^1 + 6D^1A_y^2 - 6D^2A_x^2 + 6D^2A_y^1 - 3D^1B_f^1 - 3D^1B_g^2 + 3D^2B_f^2 - 3D^2B_g^1 + \\
& 3A^1D_x^1 + 3A^1D_y^2 - 3A^2D_x^2 + 3A^2D_y^1 + B_{xx}^1 - B_{yy}^1 + 2B_{xy}^2 - 2C_{xf}^1 - 2C_{yf}^2 - \\
& 2C_{xg}^2 + 2C_{yg}^1 - 3B^1D_f^1 - 3B^1D_g^2 + 3B^2D_g^2 - 3B^2D_g^1 + 3D_{ff}^1 - 3D_{gg}^1 + 6D_{fg}^2 + \\
& 2C^1C_f^1 + 2C^1C_g^2 - 2C^2C_f^2 + 2C^2C_g^1 - C^1B_x^1 - C^1B_y^2 + C^2B_x^2 - C^2B_y^1 = 0,
\end{aligned} \tag{31}$$

$$\begin{aligned}
& 6D^2A_x^1 + 6D^2A_y^2 + 6D^1A_x^2 - 6D^1A_y^1 - 3D^2B_f^1 - 3D^2B_g^2 - 3D^1B_f^2 + 3D^1B_g^1 + \\
& 3A^2D_x^1 + 3A^2D_y^2 + 3A^1D_x^2 - 3A^1D_y^1 + B_{xx}^2 - B_{yy}^2 - 2B_{xy}^1 - 2C_{xf}^2 + 2C_{yf}^1 + \\
& 2C_{xg}^1 + 2C_{yg}^2 - 3B^2D_f^1 - 3B^2D_g^2 - 3B^1D_f^2 + 3B^1D_g^1 + 3D_{ff}^2 - 3D_{gg}^2 - 6D_{fg}^1 + \\
& 2C^2C_f^1 - 2C^2C_g^2 + 2C^1C_f^2 - 2C^1C_g^1 - C^2B_x^1 - C^2B_y^2 - C^1B_x^2 + C^1B_y^1 = 0;
\end{aligned} \tag{32}$$

3. The system of PDEs (27) and (28) has four real symmetries \mathbf{X}_1 , \mathbf{Y}_1 , \mathbf{X}_2 and \mathbf{Y}_2 with

$$\mathbf{X}_1 = \rho_1 \mathbf{X}_2 - \rho_2 \mathbf{Y}_2, \quad \mathbf{Y}_1 = \rho_1 \mathbf{Y}_2 + \rho_2 \mathbf{X}_2, \tag{33}$$

for nonconstant ρ_1 and ρ_2 and they satisfy

$$[\mathbf{X}_1, \mathbf{X}_2] - [\mathbf{Y}_1, \mathbf{Y}_2] = 0, \quad [\mathbf{X}_1, \mathbf{Y}_2] + [\mathbf{Y}_1, \mathbf{X}_2] = 0, \tag{34}$$

such that a point transformation $(x, y, f, g) \longrightarrow (X, Y, F, G)$, which brings \mathbf{X}_1 , \mathbf{Y}_1 , \mathbf{X}_2 and \mathbf{Y}_2 to their canonical form

$$\mathbf{X}_1 = \frac{\partial}{\partial F}, \quad \mathbf{Y}_1 = \frac{\partial}{\partial G}, \quad \mathbf{X}_2 = X \frac{\partial}{\partial F} + Y \frac{\partial}{\partial G}, \quad \mathbf{Y}_2 = Y \frac{\partial}{\partial F} - X \frac{\partial}{\partial G} \quad (35)$$

reduces the system to the linear form

$$F_{XX} - F_{YY} + 2G_{XY} = W_1(X, Y), \quad (36)$$

$$G_{XX} - G_{YY} - 2F_{XY} = W_2(X, Y); \quad (37)$$

4. The system of PDEs (27) and (28) has four real symmetries \mathbf{X}_1 , \mathbf{Y}_1 , \mathbf{X}_2 and \mathbf{Y}_2 with

$$\mathbf{X}_1 = \rho_1 \mathbf{X}_2 - \rho_2 \mathbf{Y}_2, \quad \mathbf{Y}_1 = \rho_1 \mathbf{Y}_2 + \rho_2 \mathbf{X}_2, \quad (38)$$

for nonconstant ρ_1 and ρ_2 and they satisfy either

$$[\mathbf{X}_1, \mathbf{X}_2] - [\mathbf{Y}_1, \mathbf{Y}_2] \neq 0 \text{ or } [\mathbf{X}_1, \mathbf{Y}_2] + [\mathbf{Y}_1, \mathbf{X}_2] \neq 0 \quad (39)$$

such that a point transformation $(x, y, f, g) \longrightarrow (X, Y, F, G)$, which brings \mathbf{X}_1 , \mathbf{Y}_1 , \mathbf{X}_2 and \mathbf{Y}_2 to their canonical form

$$\mathbf{X}_1 = \frac{\partial}{\partial F}, \quad \mathbf{Y}_1 = \frac{\partial}{\partial G}, \quad \mathbf{X}_2 = F \frac{\partial}{\partial F} + G \frac{\partial}{\partial G}, \quad \mathbf{Y}_2 = G \frac{\partial}{\partial F} - F \frac{\partial}{\partial G}, \quad (40)$$

reduces the system to the linear form

$$F_{XX} - F_{YY} + 2G_{XY} = HW_1(X, Y) - LW_2(X, Y), \quad (41)$$

$$G_{XX} - G_{YY} - 2F_{XY} = HW_2(X, Y) + LW_1(X, Y). \quad (42)$$

where

$$H = F_X + G_Y, \quad L = G_X - F_Y; \quad (43)$$

5. The system of PDEs (27) and (28) has four real symmetries \mathbf{X}_1 , \mathbf{Y}_1 , \mathbf{X}_2 and \mathbf{Y}_2 with

$$\mathbf{X}_1 \neq \rho_1 \mathbf{X}_2 - \rho_2 \mathbf{Y}_2, \quad \mathbf{Y}_1 \neq \rho_1 \mathbf{Y}_2 + \rho_2 \mathbf{X}_2, \quad (44)$$

for nonconstant ρ_1 and ρ_2 and they satisfy

$$[\mathbf{X}_1, \mathbf{X}_2] - [\mathbf{Y}_1, \mathbf{Y}_2] = 0, \quad [\mathbf{X}_1, \mathbf{Y}_2] + [\mathbf{Y}_1, \mathbf{X}_2] = 0, \quad (45)$$

such that a point transformation $(x, y, f, g) \longrightarrow (X, Y, F, G)$, which brings $\mathbf{X}_1, \mathbf{Y}_1, \mathbf{X}_2$ and \mathbf{Y}_2 to their canonical form

$$\mathbf{X}_1 = \frac{\partial}{\partial X}, \quad \mathbf{Y}_1 = \frac{\partial}{\partial Y}, \quad \mathbf{X}_2 = \frac{\partial}{\partial F}, \quad \mathbf{Y}_2 = \frac{\partial}{\partial G} \quad (46)$$

reduces the system to the system of PDEs which is at most cubic in all its first derivatives;

6. The system of PDEs (27) and (28) has four real symmetries $\mathbf{X}_1, \mathbf{Y}_1, \mathbf{X}_2$ and \mathbf{Y}_2 with

$$\mathbf{X}_1 \neq \rho_1 \mathbf{X}_2 - \rho_2 \mathbf{Y}_2, \quad \mathbf{Y}_1 \neq \rho_1 \mathbf{Y}_2 + \rho_2 \mathbf{X}_2, \quad (47)$$

for nonconstant ρ_1 and ρ_2 and they satisfy either

$$[\mathbf{X}_1, \mathbf{X}_2] - [\mathbf{Y}_1, \mathbf{Y}_2] \neq 0 \text{ or } [\mathbf{X}_1, \mathbf{Y}_2] + [\mathbf{Y}_1, \mathbf{X}_2] \neq 0 \quad (48)$$

such that a point transformation $(x, y, f, g) \longrightarrow (X, Y, F, G)$, which brings $\mathbf{X}_1, \mathbf{Y}_1, \mathbf{X}_2$ and \mathbf{Y}_2 to their canonical form

$$\mathbf{X}_1 = \frac{\partial}{\partial F}, \quad \mathbf{Y}_1 = \frac{\partial}{\partial G}, \quad \mathbf{X}_2 = X \frac{\partial}{\partial X} + Y \frac{\partial}{\partial Y} + F \frac{\partial}{\partial F} + G \frac{\partial}{\partial G}, \quad (49)$$

$$\mathbf{Y}_2 = Y \frac{\partial}{\partial X} - X \frac{\partial}{\partial Y} + G \frac{\partial}{\partial F} - F \frac{\partial}{\partial G}, \quad (50)$$

reduces the system to the linear form

$$\begin{aligned} X(F_{XX} - F_{YY} + 2G_{XY}) - Y(G_{XX} - G_{YY} - 2F_{XY}) &= a_1(H^3 - 3HL^2) - \\ a_2(3H^2L - L^3) + b_1(H^2 - L^2) - 2b_2HL + \frac{1}{3(a_1^2 + a_2^2)}\{3(a_1^2 + a_2^2) + (b_1^2 - b_2^2)a_1 + \\ 2b_1b_2a_2\}H - \frac{1}{3(a_1^2 + a_2^2)}\{2b_1b_2a_1 - a_2(b_1^2 - b_2^2)\}L + \frac{1}{3(a_1^2 + a_2^2)}(b_1a_1 + b_2a_2) + \\ \frac{1}{27(a_1^2 + a_2^2)}\{(b_1^3 - 3b_1b_2^2)(a_1^2 - a_2^2) + 2a_1a_2(3b_1^2b_2 - b_2^3)\}, \end{aligned} \quad (51)$$

$$\begin{aligned} Y(F_{XX} - F_{YY} + 2G_{XY}) + X(G_{XX} - G_{YY} - 2F_{XY}) &= a_2(H^3 - 3HL^2) + \\ a_1(3H^2L - L^3) + b_2(H^2 - L^2) + 2b_1HL + \frac{1}{3(a_1^2 + a_2^2)}\{3(a_1^2 + a_2^2) + (b_1^2 - b_2^2)a_1 + \\ 2b_1b_2a_2\}L + \frac{1}{3(a_1^2 + a_2^2)}\{2b_1b_2a_1 - a_2(b_1^2 - b_2^2)\}H + \frac{1}{3(a_1^2 + a_2^2)}(b_2a_1 - b_1a_2) + \\ \frac{1}{27(a_1^2 + a_2^2)}\{(3b_1^2b_2 - b_2^3)(a_1^2 - a_2^2) - 2(b_1^3 - 3b_1b_2^2)a_1a_2\}. \end{aligned} \quad (52)$$

The invertible real transformations

$$\tilde{X} = F + \frac{1}{3(a_1^2 + a_2^2)} \{(b_1 a_1 + b_2 a_2)X - (b_2 a_1 - b_1 a_2)Y\}, \quad (53)$$

$$\tilde{Y} = G + \frac{1}{3(a_1^2 + a_2^2)} \{(b_1 a_1 + b_2 a_2)Y + (b_2 a_1 - b_1 a_2)X\}, \quad (54)$$

$$\begin{aligned} \tilde{F} = & \frac{1}{2}(F^2 - G^2) + \frac{1}{3(a_1^2 + a_2^2)} [\{(b_1 a_1 + b_2 a_2)X - (b_2 a_1 - b_1 a_2)Y\}F - \\ & \{(b_1 a_1 + b_2 a_2)Y + (b_2 a_1 - b_1 a_2)X\}G] + \frac{1}{18(a_1^2 + a_2^2)^2} [\{(b_1^2 - b_2^2)(a_1^2 - a_2^2) + \\ & 4b_1 b_2 a_1 a_2\}(X^2 - Y^2) - 2XY\{(2b_1 b_2(a_1^2 - a_2^2) - 2a_1 a_2(b_1^2 - b_2^2))\}] + \\ & \frac{1}{2(a_1^2 + a_2^2)} \{a_1(X^2 - Y^2) + 2a_2 XY\}, \end{aligned} \quad (55)$$

$$\begin{aligned} \tilde{G} = & FG + \frac{1}{3(a_1^2 + a_2^2)} [\{(b_1 a_1 + b_2 a_2)X - (b_2 a_1 - b_1 a_2)Y\}G + \\ & \{(b_1 a_1 + b_2 a_2)Y + (b_2 a_1 - b_1 a_2)X\}F] + \frac{1}{18(a_1^2 + a_2^2)^2} [2XY\{(b_1^2 - b_2^2)(a_1^2 - a_2^2) + \\ & 4b_1 b_2 a_1 a_2\} - (X^2 - Y^2)\{2b_1 b_2(a_1^2 - a_2^2) - 2a_1 a_2(b_1^2 - b_2^2)\}] + \\ & \frac{1}{2(a_1^2 + a_2^2)} \{2XY a_1 - a_2(X^2 - Y^2)\}. \end{aligned} \quad (56)$$

transform (51) and (52) into the system of linear PDEs

$$\tilde{F}_{\tilde{X}\tilde{X}} - \tilde{F}_{\tilde{Y}\tilde{Y}} + 2\tilde{G}_{\tilde{X}\tilde{Y}} = 0, \quad (57)$$

$$\tilde{G}_{\tilde{X}\tilde{X}} - \tilde{G}_{\tilde{Y}\tilde{Y}} - 2\tilde{F}_{\tilde{X}\tilde{Y}} = 0. \quad (58)$$

Examples. Now we discuss some illustrative examples.

1. Consider a nonlinear CODE of the form

$$u'' + 3uu' + u^3 = 0, \quad (59)$$

which is linearizable as it satisfies the Lie complex conditions. The set of two non-commuting CLSs is

$$\mathbf{Z}_1 = \frac{\partial}{\partial z}, \quad \mathbf{Z}_2 = z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u}. \quad (60)$$

Note that $\mathbf{Z}_1 \neq \rho(z, u)\mathbf{Z}_2$. We invoke condition six of Theorem 1 to find a linearization transformation. The complex point transformation that reduces the symmetries (60) to their canonical form is

$$Z = \frac{1}{u}, \quad U = z + \frac{1}{u}, \quad (61)$$

and (59) reduces to

$$ZU'' = -U'^3 + 6U'^2 - 11U' + 6, \quad (62)$$

by means of the transformation (61). Equation (59) linearizes to $\tilde{U}'' = 0$ via the complex transformations (25) by placing $a = -1$, $b = 6$ in equation (34) in P-I. That is

$$\tilde{Z} = z - \frac{1}{u}, \quad \tilde{U} = \frac{z^2}{2} - \frac{z}{u}. \quad (63)$$

It may be seen that the above transformation is a proper complex transformation thus one does not get a contradiction as was in P-I. Further, the procedure that seems to be analogous to analytic continuation do not arise here. The system of PDEs corresponding to (59) is

$$f_{xx} - f_{yy} + 2g_{xy} = -3(fh - gl) - (f^3 - 3fg^2), \quad (64)$$

$$g_{xx} - g_{yy} - 2f_{xy} = -3(gh + fl) - (3f^2g - g^3), \quad (65)$$

which admits the RLSs

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{Y}_1 = \frac{\partial}{\partial y}, \quad (66)$$

$$\mathbf{X}_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g}, \quad (67)$$

$$\mathbf{Y}_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - g\frac{\partial}{\partial f} + f\frac{\partial}{\partial g}. \quad (68)$$

These generators satisfy (47) and (49). The transformation

$$X = \frac{f}{f^2 + g^2}, \quad Y = \frac{-g}{f^2 + g^2}, \quad (69)$$

$$F = x + \frac{f}{f^2 + g^2}, \quad G = y - \frac{g}{f^2 + g^2}, \quad (70)$$

transforms (64) and (65) into

$$X(F_{XX} - F_{YY} + 2G_{XY}) - Y(G_{XX} - G_{YY} - 2F_{XY}) = -H^3 + 3HL^2 - 6H^2 - 11H + 6, \quad (71)$$

$$X(G_{XX} - G_{YY} - 2F_{XY}) + Y(F_{XX} - F_{YY} + 2G_{XY}) = L^3 - 3H^2L + 12HL - 11L, \quad (72)$$

where H and L are given by (43). The above system can further be reduced into very simplified linear PDEs by using

$$\tilde{X} = x - \frac{f}{f^2 + g^2}, \quad \tilde{Y} = y + \frac{g}{f^2 + g^2}, \quad (73)$$

$$\tilde{F} = \frac{1}{2}(x^2 - y^2) - \frac{1}{f^2 + g^2}(xf + yg), \quad (74)$$

$$\tilde{G} = xy - \frac{1}{f^2 + g^2}(yf - xg). \quad (75)$$

The linear PDEs are

$$\tilde{F}_{\tilde{X}\tilde{X}} - \tilde{F}_{\tilde{Y}\tilde{Y}} + 2\tilde{G}_{\tilde{X}\tilde{Y}} = 0, \quad (76)$$

$$\tilde{G}_{\tilde{X}\tilde{X}} - \tilde{G}_{\tilde{Y}\tilde{Y}} - 2\tilde{F}_{\tilde{X}\tilde{Y}} = 0. \quad (77)$$

2. Consider a second order nonlinear CODE with an arbitrary function $w(z)$

$$uu'' = u'^2 + w(z)u^2. \quad (78)$$

The above CODE admits two CLSs of the form

$$\mathbf{Z}_1 = zu \frac{\partial}{\partial u}, \quad \mathbf{Z}_2 = u \frac{\partial}{\partial u}, \quad (79)$$

and thus

$$[\mathbf{Z}_1, \mathbf{Z}_2] = 0 \text{ and } \mathbf{Z}_2 = \frac{1}{z} \mathbf{Z}_1. \quad (80)$$

By using the complex transformation

$$Z = \frac{1}{z} \text{ and } U = \frac{1}{z} \log u \quad (81)$$

(78) can be reduced into a linear CODE

$$U'' = \frac{1}{Z^3} w\left(\frac{1}{Z}\right). \quad (82)$$

The system of PDEs corresponding to (78) are

$$f(f_{xx} - f_{yy} + 2g_{xy}) - g(g_{xx} - g_{yy} - 2f_{xy}) = h^2 - l^2 + w_1(f^2 - g^2) - 2fgw_2, \quad (83)$$

$$f(g_{xx} - g_{yy} - 2f_{xy}) + (f_{xx} - f_{yy} + 2g_{xy})g = 2hl + 2w_1fg + w_2(f^2 - g^2). \quad (84)$$

Invoking the transformation

$$X = \frac{x}{x^2 + y^2}, \quad Y = \frac{-y}{x^2 + y^2} \quad (85)$$

$$F = \frac{1}{x^2 + y^2} \left[\frac{1}{2} x \ln(f^2 + g^2) + y \tan^{-1}\left(\frac{g}{f}\right) \right], \quad (86)$$

$$G = \frac{1}{x^2 + y^2} \left[x \tan^{-1}\left(\frac{g}{f}\right) - \frac{y}{2} \ln(f^2 + g^2) \right]. \quad (87)$$

Equations (83) and (84) reduce to the linear system of PDEs

$$F_{XX} - F_{YY} + 2G_{XY} = \frac{1}{X^2 + Y^2} [(X^3 - 3XY^2)w_1 - (Y^3 - 3X^2Y)w_2], \quad (88)$$

$$G_{XX} - G_{YY} - 2F_{XY} = \frac{1}{X^2 + Y^2} [(X^3 - 3XY^2)w_2 + (Y^3 - 3X^2Y)w_1], \quad (89)$$

where

$$w_1 = w_1\left(\frac{X}{X^2 + Y^2}, \frac{-Y}{X^2 + Y^2}\right), \quad w_2 = w_2\left(\frac{X}{X^2 + Y^2}, \frac{-Y}{X^2 + Y^2}\right). \quad (90)$$

3. Consider the nonlinear CODE

$$u'' = \frac{1}{z}(u' + u'^3). \quad (91)$$

This equation has CLSs

$$\mathbf{Z}_1 = \frac{1}{z} \frac{\partial}{\partial z}, \quad \mathbf{Z}_2 = \frac{u}{z} \frac{\partial}{\partial z}. \quad (92)$$

The complex transformation

$$U = \frac{1}{2}z^2 \text{ and } Z = u, \quad (93)$$

transforms (91) into a linear CODE

$$U'' + 1 = 0. \quad (94)$$

The system of PDEs corresponding to (91) is

$$f_{xx} - f_{yy} + 2g_{xy} = \frac{1}{x^2 + y^2} [xh(1 + h^2 - 3l^2) + yl(1 - l^2 + 3h^2)], \quad (95)$$

$$g_{xx} - g_{yy} - 2f_{xy} = \frac{1}{x^2 + y^2} [xl(1 - l^2 + 3h^2) + yh(1 + h^2 - 3l^2)], \quad (96)$$

and admits CLSs

$$\mathbf{X}_1 = \frac{x}{x^2 + y^2} \frac{\partial}{\partial x} - \frac{y}{x^2 + y^2} \frac{\partial}{\partial y}, \mathbf{Y}_1 = \frac{-y}{x^2 + y^2} \frac{\partial}{\partial x} - \frac{x}{x^2 + y^2} \frac{\partial}{\partial y}, \quad (97)$$

$$\mathbf{X}_2 = \frac{fx}{x^2 + y^2} \frac{\partial}{\partial x} - \frac{fy}{x^2 + y^2} \frac{\partial}{\partial y} + \frac{yg}{x^2 + y^2} \frac{\partial}{\partial x} - \frac{gx}{x^2 + y^2} \frac{\partial}{\partial y}, \quad (98)$$

$$\mathbf{Y}_2 = \frac{gx}{x^2 + y^2} \frac{\partial}{\partial x} - \frac{gy}{x^2 + y^2} \frac{\partial}{\partial y} - \frac{fy}{x^2 + y^2} \frac{\partial}{\partial x} - \frac{fx}{x^2 + y^2} \frac{\partial}{\partial y}. \quad (99)$$

The above RLSs correspond to the condition five of Theorem 1. The real transformation that linearize system (95) and (96) is

$$X = f, Y = g, \quad F = \frac{1}{2}(x^2 - y^2), G = xy. \quad (100)$$

The linearized system of PDEs is

$$F_{XX} - F_{YY} + 2G_{XY} + 1 = 0, \quad (101)$$

$$G_{XX} - G_{YY} - 2F_{XY} = 0. \quad (102)$$

4. Consider the nonlinear CODE

$$u'' = 1 + (u' - z)^2 w(2u - z^2) \quad (103)$$

which admits the CLSs

$$\mathbf{Z}_1 = \frac{\partial}{\partial z} + z \frac{\partial}{\partial u}, \quad \mathbf{Z}_2 = z \frac{\partial}{\partial z} + z^2 \frac{\partial}{\partial u}. \quad (104)$$

Equation (103) reduces to a linear CODE by using the complex transformation

$$Z = 2u - z^2 \text{ and } U = z. \quad (105)$$

It becomes

$$U'' = \frac{-1}{2} U' w(Z). \quad (106)$$

The system of PDEs corresponding to (103) is

$$f_{xx} - f_{yy} + 2g_{xy} = 1 + \{(h-x)^2 - (l-y)^2\}w_1 - 2(h-x)(l-y)w_2, \quad (107)$$

$$g_{xx} - g_{yy} - 2f_{xy} = 2(h-x)(l-y)w_1 + \{(h-x)^2 - (l-y)^2\}w_2, \quad (108)$$

which admits RLSs

$$\mathbf{X}_1 = \frac{\partial}{\partial x} + x \frac{\partial}{\partial f} + y \frac{\partial}{\partial g}, \quad \mathbf{Y}_1 = x \frac{\partial}{\partial g} - y \frac{\partial}{\partial f} - \frac{\partial}{\partial y}, \quad (109)$$

$$\mathbf{X}_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + (x^2 - y^2) \frac{\partial}{\partial f} + 2xy \frac{\partial}{\partial g}, \quad (110)$$

$$\mathbf{Y}_2 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + 2xy \frac{\partial}{\partial f} - (x^2 - y^2) \frac{\partial}{\partial g}, \quad (111)$$

which satisfy condition six of Theorem 1. The transformation

$$X = 2f - x^2 + y^2, \quad Y = 2g - 2xy, \quad (112)$$

$$F = x, \quad G = y. \quad (113)$$

yields system of linear PDEs

$$F_{XX} - F_{YY} + 2G_{XY} = -\frac{1}{2}(Hw_1 - Lw_2), \quad (114)$$

$$G_{XX} - G_{YY} - 2F_{XY} = -\frac{1}{2}(Hw_2 + Lw_1), \quad (115)$$

where

$$w(z) = w_1 + iw_2, \quad H = F_X + G_Y, \quad L = G_X - F_Y. \quad (116)$$

4 Conclusion

Finding exact solutions of nonlinear DEs is difficult. Linearization of DEs is a method to convert the original nonlinear DE to an equivalent linear DE so that exact solutions can be constructed directly. But it requires the existence and construction of such transformations (point, tangent, contact or complex) that transforms nonlinear DEs into linear DEs. For scalar ODEs Lie provided a nontrivial construction of point transformations via symmetries, but it is difficult to extend the Lie conditions for systems of ODEs and for PDEs (see e.g., [5, 6, 13], [15 – 19]). Finding transformations that map systems of nonlinear ODEs and PDEs into linear systems is not only tedious but also highly nontrivial.

We have applied CLS analysis to find non-trivial ways of *reducing order, linearizing and solving* certain systems of ODEs and PDEs. Several known results of classical symmetry method were studied in the complex domain. Various results for systems of PDEs were obtained, e.g. the use of CLS analysis to study CLSs for CODEs and respective real Lie symmetries for systems of PDEs corresponding to CODEs, discussed in [1]. Variational problems involving complex Lagrangians of CODEs results in Lagrangians of systems of PDEs corresponding to those CODEs and double reduction in order of CODEs via complex Noether symmetries, which are studied in [2]. Further, variational problems for r-CODEs and their use in constructing Lagrangians for systems of RODEs are discussed in [4].

We have looked at linearizability criteria for second order CODEs and obtained analogous Lie conditions for CODEs and applied them to the linearization of systems of nonlinear PDEs in this work. The corresponding statement for a system of two PDEs (in fact four i.e. by incorporating linear CR-equations) associated to a second order CODE was presented. Examples were given for the linearization of systems of PDEs including the construction of the point linearizing transformations.

In this paper the complex function is a function on the complex plane that requires CR-equations to hold for complex differentiability. Thus, we mapped complex solutions of CODEs into complex solutions via complex transformations. Since, CR-equations are linear this makes no difference in the linearization of systems of PDEs corresponding to CODEs except that all the symmetries are analytic i.e. their coefficients satisfy CR-equations. These form the conformal subalgebra admitted by systems of PDEs. It is hoped that further classification of systems of PDEs in [1] with respect to

conformal algebras admitted by them can be achieved.

We have seen that linearization theorems can be carried over from the real case to the complex case with remarkable results for the linearization of systems of PDEs. Thus we were able to linearize those systems of nonlinear PDEs that correspond to some CODEs. There are several possible ways of using CLS analysis. It would be of great interest to extend the geometric proof of linearization for RODEs [10, 20] to CODEs. The analysis can also be used for classification of those systems of nonlinear PDEs that correspond to CODEs. An extension of the Lie linearization results for systems of quadratically and cubically semi-linear CODEs in [19, 20], in the complex domain may yield useful and highly nontrivial results for systems of PDEs. Further, results of linearization for third and fourth order CODEs may be obtained by using results for third and fourth order RODEs in [18, 21, 22].

It is expected that the Lie criteria for linearization of two CODEs can be found in a similar way to the above, which can then be used to linearize four nonlinear PDEs corresponding to a system of two CODEs. A system of two second order RODEs admits 5, 6, 7, 8 or 15 RLSs and the maximal symmetry algebra is $sl(4, \mathbb{R})$ for the simplest system [27]. Thus, it may be argued that the system of two second order CODEs also admit 5, 6, 7, 8 or 15 CLSs. These result in 10, 12, 14, 16 or 30 RLSs which form the subalgebra of symmetries admitted by four second order PDEs that correspond to two second order CODEs. Further, maximal symmetry algebra $sl(4, \mathbb{C})$ is admitted by the simplest systems of two second order CODEs.

It is also hoped that Penrose's "complex magic" can be further extended to "*hypercomplex magic*" by using hypercomplex variables (e.g. quaternions, octonions, Clifford or Grassman variables) that should provide hypercomplex Lie symmetries. These can then be used to reduce the order, linearize and solve a large class of systems of four PDEs by using a scalar ODE in hypercomplex variables. It is further expected that the non-commutative behavior of hypercomplex variables will yield a non-trivial generalization of Lie's work. Further, use of the hypercomplex Lie symmetries in geometric calculus [9] may result in non-trivial physical implications.

In P-I, we used CLS analysis to construct linearizability criteria for certain systems of nonlinear RODEs by considering r-CODEs. An r-CODE is a complex ODE in a single complex function of one real variable that yields a system of two RODEs. The linearization of r-CODEs directly gives linearizability criteria for systems of RODEs corresponding to those r-CODEs [3]. Again, the use of hypercomplex variables should yield linearizability criteria

for systems of four ODEs. The possibilities to extend Lie's programme of linearizing nonlinear differential equations, and determining which equations can be so reduced, seem endless. We have only touched the tip of the iceberg.

Acknowledgments

SA is most grateful to NUST and DECMA in providing financial assistance for his stay at Wits University, South Africa, where this work was done. AQ acknowledges the School of Computational and Applied Mathematics (DECMA) for funding his stay at the university.

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